Area laws for one- and two-dimensional systems

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Abstract

The area-law conjecture states that the entanglement entropy of the ground state of a gapped local Hamiltonian only scales with the boundary area of the bipartition. It is one of the central questions in quantum Hamiltonian complexity due to its profound implications for classical simulability of quantum many-body systems. Since the first proof for the 1D case by Hastings [1], new mathematical tools have been developed to simplify the proofs specifically for frustration-free systems in hoping to approach higher-dimensional area laws. In this report, we present an overview of the Chebyshev-based Approximate Ground State Projector technique, which has enjoyed successes in proving 1D and 2D area laws. We review the seminal papers [2–4] that successfully employed this technique.

1 Introduction

Ground states of local Hamiltonians are the central object of study in quantum many-body physics. Quantum entanglement gives rise to an exponentially large Hilbert space, which is the fundamental obstacle in simulating quantum systems on a classical computer. What are the conditions under which these ground states can be described efficiently, and when are quantum systems efficiently simulatable on a classical computer? A number of many-body physics algorithms based on tensor networks [5] have been developed, where one hopes to find efficient descriptions of quantum states. These algorithms all require that the quantum states have small entanglement. Area laws provide a tool for bounding the entanglement that ground states of a quantum system can exhibit.

Definition 1.1 (Entanglement entropy)**.** *If ρ_{AB} is a state of a bipartite system, then its entanglement entropy is defined as the von Neumann entropy of the reduced density matrix of its subsystems, i.e., abusing notation, $S(\rho_{AB}) = S(\mathrm{Tr}_A[\rho_{AB}]) = S(\mathrm{Tr}_B[\rho_{AB}])$.*

Definition 1.2 (k-local lattice Hamiltonians)**.** *Consider n particles with local dimension d (e.g. $d = 2$ for qubits) that are positioned on the sites of a regular D-dimensional lattice with an underlying Hilbert space $(\mathbb{C}^d)^{\otimes n}$. On this lattice we consider a k-local Hamiltonian $H = \sum_i h_i$, where each h_i acts on at most k neighboring particles of the lattice and $0 \leq h_i \leq 1$.*

A state of a lattice spin system is said to satisfy an area law if its entanglement entropy with respect to any bipartition scales with the size of its boundary.

Conjecture 1.3 (The area-law conjecture)**.** *The ground states of any constant-gapped and locally interacting lattice quantum systems satisfy the area law.*

This is a much stronger statement than the trivial bound (volume laws), which says that the entanglement entropy is proportional to the number of particles in the subsystems. An intuitive reason to believe in this conjecture is that one might expect the entanglement is only generated by the local interactions near the boundary. However, it should be noted that a more general statement of the area-law conjecture for quantum systems defined on graphs has been disproved via a counter-example in [6].

Conjecture 1.3 has been proved for *all* 1D systems by Hastings [1], but the higher dimensional cases remain an open problem. The proof in [1] uses the Lieb-Robinson bound and sophisticated Fourier analysis to achieve a very rough upper bound that does not scale up to higher dimensions. In this paper, we focus on a new technique that convolves ideas from combinatorics and approximation theory: Approximation Ground State Projector (AGSP) [2, 3, 7]. AGSP provides hopes to improving the entanglement bound and scaling up. A timeline of the development of AGSP-based area law proofs presented to this paper is as follows. Ref. [7] provides the first AGSP-based area law proof for 1D frustration-free (FF) systems (defined below). Ref. [2] exponentially improves the entanglement bound over those in [1, 7] on FF systems. Ref. [3] proves the area law for general 1D systems, their bound is also believed to be tight in terms of the spectral gap. Most recently, ref. [4] provides the first proof for 2D FF systems with an extra assumption called "locally gapped".

Definition 1.4 (Frustration-free Hamiltonians)**.** *A system is frustration-free when every ground state* $|\Omega\rangle \in V_{gs}$ *minimizes the energy of each local term* h_i *separately. That is,* $h_i|\Omega\rangle = 0$ *for any* i*.*

Examples of such systems include the AKLT model, spin 1/2 ferromagnetic XXZ chain, and Kitaev's toric code. These systems are somewhat more "classical" and working with them is often considered the first step towards proving area laws on general (frustrated) systems. This intuition can be seen via an analogy to the classical constraint satisfaction problems (CSP). An FF Hamiltonian is thus in analogy to a CSP where all constraints can be satisfied simultaneously. To make the problem even more CSP-like, one can introduce an auxiliary Hamiltonian in which every h_i is replaced by a projector Q_i whose null space coincides with the null space of h_i. The auxiliary Hamiltonian $\hat{H} := \sum_i Q_i$ and the original Hamiltonian $H = \sum_i h_i$ share the same ground space. Furthermore, since $0 \leq h_i \leq Q_i$, the spectral gaps are related by $\gamma(\hat{H}) \geq \gamma(H)$. Thus, from now on we will assume H is a sum of local projectors Q_i. In addition, we denote the local projector on the ground space of Q_i as $P_i = 1 - Q_i$.

The rest of this report will mostly discuss the applications of AGSP on frustration-free spin systems. Section 2 reviews some important tools for proving area laws, including AGSP, the detectability lemma, and Chebyshev polynomials. Then, we provide an overview of the proofs in [2, 3] and [4] for 1D and 2D FF systems in Section 3 and Section 4. Finally, we discuss open questions and other research directions related to area laws in Section 5

2 Preliminary tools

2.1 Approximate ground state projector

A fruitful approach to proving an area law is to find a projector K that approximates the ground space projector $\Pi_{gs} := |\Omega\rangle\langle\Omega|$ whose Schmidt rank across the cut can be controlled. The Schmidt decomposition across a bipartition of a pure state is defined as $|\Psi\rangle_{AB} = \sum_{i=1}^{k} \lambda_i |\psi_i\rangle_A |\phi_i\rangle_B$, where each set $\{|\psi_i\rangle\}_i$ or $\{|\phi_i\rangle\}_i$ forms an orthonormal set in the associated Hilbert space, $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k > 0$, and $\sum_i \lambda_i^2 = 1$. Then, the Schmidt rank of $|\Psi\rangle$ is $SR(\Psi) = k$. Similarly, the Schmidt rank of an operator O is $SR(O) := \max_\psi SR(O\psi)/SR(\psi)$.

Definition 2.1 (Approximate ground state projector (AGSP))**.** *An operator* K *is a* (D, Δ)*-AGSP if it satisfies the following properties*

(a) (Ground space invariance) $K|\Omega\rangle = |\Omega\rangle$ *for any* $|\Omega\rangle \in V_{gs}$

(b) (Shrinking factor) $\|K|\Omega_\perp\rangle\|^2 \leq \Delta$ *and also* $K|\Omega_\perp\rangle \in V_{gs}^\perp$ *for any normalized state* $|\Omega_\perp\rangle$ *in* $V_{gs}^\perp$

(c) (Entanglement) $SR(K) \leq D$

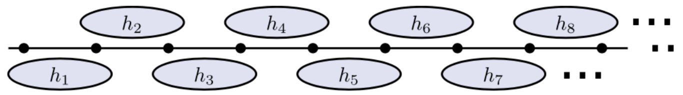


Figure 1: [9] Decomposing the local terms of a 1D Hamiltonian $H = \sum_i h_i$ with 2-local, nearest-neighbor interactions into two layers an even layer and an odd layer. Here, $h_i = Q_i$ and $P_i = 1 - Q_i$. So $\Pi_{even} = \prod_{i \text{ even}} P_i$, $\Pi_{odd} = \prod_{i \text{ odd}} P_i$ and $\text{DL}(H) = \Pi_{even}\Pi_{odd}$.

The key idea is to first find a product state $|\Psi\rangle = |\psi\rangle_A \otimes |\phi\rangle_B$ which has a non-trivial overlap with Ω. If D and Δ are small, we can use K and $|\Psi\rangle$ to generate a state with small Schmidt rank that approximates $|\Omega\rangle$. In particular, we will iteratively apply K on $|\Psi\rangle$ and normalizing the resultant state (that is $|v_\ell\rangle = \frac{K^\ell|\Psi\rangle}{\|K^\ell|\Psi\rangle\|}$ with $|v_0\rangle$ being $|\Psi\rangle$). Observe that $SR(|v_\ell\rangle) \leq D^\ell$, thus we hope to find a good AGSP, one that 'shrinks' the excited space $V_{gs}^\perp$ fast enough such that only a few applications of K are needed. For example, if $K = \Pi_{gs}$ (such that it is a $(SR(\Omega), 0)$-AGSP), then one application of K immediately yields $|\Omega\rangle$. This intuition will be made more rigorous when we present the work of [2] in Section 3.

2.2 The detectability lemma

One of the original approaches for constructing AGSP in FF systems is the detectability lemma (DL) [8]. The DL operator is constructed by first arranging the Hamiltonian terms into L "layers" $T_1, \ldots, T_L$ whose terms are pairwise commuting. For each layer, define the *layer projector* to be $\Pi_\ell := \prod_{i \in T_\ell} P_i$. Then, the DL operator is defined as

$$\text{DL}(H) := \prod_{\ell=1}^{L} \prod_{i \in T_\ell} P_i. \tag{1}$$

For instance, the DL operator in a 2-local 1D Hamiltonian is visualized in Figure 1.

The detectability lemma provides a bound on the shrinking factor of the DL operator.

Lemma 2.2 (Detectability lemma (improved) [9]). *For any state $|\Omega_\perp\rangle$ orthogonal to the ground space of 2-local 1D Hamiltonian H of spectral gap γ,*

$$\|\text{DL}(H)\left|\psi^\perp\right\rangle\|^2 \leq (\gamma/4 + 1)^{-1}.$$

One of the great benefits of using the DL is that the all projections in a given layer commute, and, hence, the analysis becomes almost classical. Aharonov et al. [7] use the DL operator to construct an alternative proof to Hastings [1] on FF systems, but the entanglement bound does not improve. One of reasons is because the DL operator is a polynomial of degree n in the local terms, making its Schmidt rank too large.

2.3 Chebyshev polynomials

Chebyshev polynomials are a powerful tool in approximation theory and provide a way to improve the polynomial construction of AGSPs. For instance, [2] use Chebyshev polynomials and DL operator to improve the entanglement bound in 1D FF systems. Later works [3, 4], which do not rely on the DL, also use them to construct AGSPs.

Definition 2.3 (Chebyshev polynomials of the first kind). *The Chebyshev polynomial family $\{T_n(x)\}$, where n is the polynomial degree can be equivalently defined as follows*

(a) (Ordinary differential equations) $T_n(x)$ is the solution of

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0.$$

(b) (Trigonometric form) If $|x| \leq 1$, then

$$T_n(x) \overset{def}{=} \cos\left(n \cos^{-1} x\right).$$

(c) (Recursive form) $T_0(x) = 1$, $T_1(x) = x$, and

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

The Chebyshev polynomials are used to approximate the step function $f_{step}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ otherwise} \end{cases}$. To this end, one needs to define a modified Chebyshev polynomial. For $\eta \in (0, 1)$ and $s > 0$, define $T_{\eta,s} : [0, 1] \to \mathbb{R}$ as

$$T_{\eta,s}(x) \overset{\text{def}}{=} \frac{T_{\lceil s \rceil}\left(\frac{2(1-x)}{1-\eta} - 1\right)}{T_{\lceil s \rceil}\left(\frac{2}{1-\eta} - 1\right)}. \tag{2}$$

The following lemma [2] states that $T_{\eta,s}$ is an approximator of f_{step}.

Lemma 2.4. *$T_{\eta,s}(0) = 1$ and $|T_{\eta,s}(x)| \leq 2e^{-2s\sqrt{\eta}}$ for $\eta \leq x \leq 1$.*

3 1D area laws

3.1 Frustration-free systems

We present an overview of the proof of the area law for 1D frustration-free (FF) systems by Arad-Landau-Vazirani [2].

Theorem 3.1 (Area law for 1D frustration-free systems)**.** *Consider a gapped, frustration-free 1D chain of locally interacting particles. Let d be the dimension of each particle and $\gamma = O(1)$ be the spectral gap of the system and define $X = \frac{\log d}{\gamma}$. The entanglement entropy across any cut in the chain is bounded by*

$$S \leq O(1) \cdot X^3 \log^8 X.$$

The proof finds a good AGSP using a Chebyshev polynomials in terms of the local terms Q_i. First, the following lemma quantifies the connection between the existence of a good AGSP and a bound on the entanglement entropy.

Lemma 3.2 (Entanglement entropy bound via good AGSP)**.** *If there exists a (D, Δ)-AGSP K with $D \cdot \Delta < 1/2$, then the ground state entanglement entropy is bounded by*

$$S \leq O(1) \cdot \log D.$$

Proof sketch. The proof has two steps. The first step shows that there exists a product state $|L, R\rangle$ such that $\langle\Omega|L, R\rangle \geq 1/\sqrt{2D}$. This state can be found by starting from a product state $|\phi'\rangle$ who does not satisfy this inequality. Then it can be shown that a Schmidt vector of $\frac{K|\phi'\rangle}{\|K|\phi'\rangle\|}$ has strictly larger overlap with $|\Omega\rangle$ than $|\phi'\rangle$ does. Thus, iteratively applying this procedure produces a product state whose overlap with Ω is arbitrarily close to $1/\sqrt{2D}$. From here, the second step applies K on $|L, R\rangle$ a few times to constructs a state with low Schmidt rank and high overlap with $|\Omega\rangle$. Applying the Young-Eckart theorem, one can then show that only a few Schmidt coefficients of $|\Omega\rangle$ contribute significantly to the entanglement entropy. $\square$

Armed with the above lemma, ref. [2] finds a good AGSP with $\log D \leq O(1) \cdot X^3 \log^8 X$ stated below, whereby establishing Theorem 3.1.

Lemma 3.3 (The diluting lemma)**.** *There exists a* (D, Δ)*-AGSP with* $D \cdot \Delta < 1/2$ *and* $\log D \leq O(1) \cdot X^3 \log^8 X$ *where* $X = \frac{\log d}{\gamma}$.

Proof sketch. The proof follows the steps below

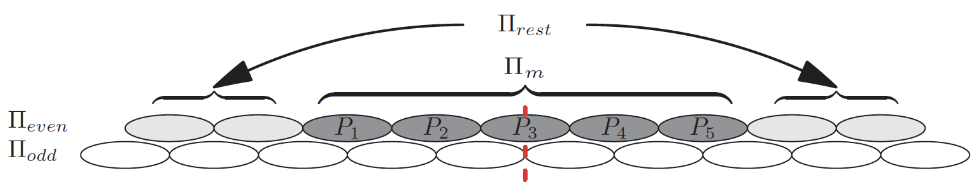
1. Assumptions: the Hamiltonian terms are 2-local, thus $\{P_i\}_i$ can be partitioned into two commuting "layers" Π_{odd} and Π_{even}; in addition assume the cut intersects Π_{even} at particles i^*, i^*+1.

2. Observation: the naive DL operator $\Pi_{even}\Pi_{odd}$ does not work because it has $D_0 = d^2$ (due to the projector P_i that intersects with the cut) and $\Delta_0 \leq \frac{1}{1+\gamma/4}$ (improved DL [9]). In general $D_0 \cdot \Delta_0 > 1$.

3. Idea 1 (coarse-graining): one can fuse k adjacent particles (making them a single particle of dimension d^k) and set the new projector P_i' to be the projector onto the common ground space of the original $2k$ adjacent particles corresponding to the coarse-grained particles at i and $i+1$. It can be shown that the DL operator of this coarse-grained Hamiltonian is $(D', \Delta') = (D_0^k, \Delta_0^k)$ (Claim IV.7 [2]). This trick can yield $D' \cdot \Delta' < 1/2$ if $D_0 \cdot \Delta_0 < 1$, but this condition generally does not hold.

4. Goal: instead of letting D grow trivially as above (via coarse-graining or simply exponentiating K^ℓ), we seek a $(D_1 D_2, \Delta)$-AGSP A such that A^ℓ is a $(D_1 D_2^\ell, \Delta^\ell)$-AGSP and $D_2 \cdot \Delta < 1/2$. If we can find such a projector A, then A^{ℓ_0} is a good AGSP for any $\ell_0 > \lceil \log D_1 \rceil$.

5. Idea 2: we modify the DL operator. We split Π_{even} into Π_m, composing of m projectors around the cut (called the middle region), and Π_{rest} (the remaining projectors in Π_{even}), then approximate Π_m by some $\hat{\Pi}_m$. So that our AGSP is $A = \hat{\Pi}_m \Pi_{rest} \Pi_{odd}$. The idea is that only m projectors (m to be determined) around the cut contribute significantly to the Schmidt rank, thus we hope to approximate Π_m by some $\hat{\Pi}_m$ whose Schmidt rank can be better controlled.



6. Fact (controlling the SR of one cut by another): in a 1D system, if r_i and r_j are the SR of $|\Psi\rangle$ that correspond to two cuts between particles $i, i+1$ and $j, j+1$, respectively, then $d^{-|i-j|} r_j \leq r_i \leq d^{|i-j|} r_j$ (recall that d is the Hilbert space dimension of each particle).

7. Idea 3: we choose $\hat{\Pi}_m$ to be a polynomial in Q_i's (only the projectors in the middle region), we bound the SR of each term in the polynomial with respect to a cut (different terms might have different cuts, see Claim IV.6 [2]). Then we lift these bounds to the cut at i^*, i^*+1. Due to the fact in step 6, the SR of each term in $\hat{\Pi}_m$ with respect to this cut is larger by a factor of at most $d^{\lceil m/2 \rceil} < D_0^m$. This factor will play the role of D_1 mentioned in step 4 above.
   Why do we want to construct $\hat{\Pi}_m$ this way? What properties do we want the polynomial to have?

   - First, if we merely use $\Pi_m = \prod_{i=1}^m P_i = \prod_{i=1}^m (1 - Q_i)$ (which is a degree-m polynomial in terms of the projectors), the SR of A^ℓ grows as D_0^ℓ which is too fast as stated earlier.
   - We want to bound the SR of $A^\ell = (\hat{\Pi}_m \Pi_{rest} \Pi_{odd})^\ell$, which is a polynomial of the middle-region projectors. We can bound the SR of each term in the polynomial separately using any cut of choice, then sum these SR bounds to obtain a SR bound for A^ℓ. The resultant "amortized" bound can be better than the trivial bound where every term uses the same cut.

- Therefore, we want $\hat{\Pi}_m$ to be a low-degree polynomial approximating Π_m.

8. Idea 4 (Π_m as a step function): observe that if we define $\mathbb{N} = \sum_{i=1}^{m} Q_i$ and $f_{step}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ otherwise} \end{cases}$, then $f_{step}(\mathbb{N}) = \Pi_m$. This follows from that fact that Q_i are commuting projectors, so $\mathbb{N}$ has non-negative integer eigenvalues and its ground space is exactly what Π_m projects onto. Our task is now clear: we need to approximate f_{step} by a low-degree polynomials.

9. Idea 5 (Chebyshev polynomials): we choose $\hat{\Pi}_m = [C_m(\mathbb{N})]^q$ where $C_m(x)$ is a polynomial defined in terms of the degree-$\sqrt{m}$ Chebyshev polynomial (Section 2.3) and q is to be chosen. C_m is designed such that $C_m(0) = 1$ and $|C_m(x)|^2 < 1/9$ for any $1 \leq x \leq m$. Hence, $\hat{\Pi}_m$ is a degree-$q\sqrt{m}$ polynomial of Q_i's, which is a significant improvement over the trivial degree m. The shrinking factor of $A = \hat{\Pi}_m \Pi_{rest} \Pi_{odd}$ can also be easily bounded by $\Delta_0 + (1/9)^q$.

10. Bounding the SR of A^ℓ: this is the main technical part of the proof, but the idea is as explained in step 7 above. It turns out that A^ℓ is a $(D_1 D_2^\ell, \Delta'^\ell)$, where $D_1 = D_0^m$, $D_2 = 20(q\sqrt{m})^{3/2} 2^{\frac{1}{2}\log^2 q\sqrt{m}} D_0^{q/\sqrt{m}}$, and $\Delta' = \Delta_0 + (1/9)^q$.

11. At this point, we need to ensure $D_2 \cdot \Delta' < 1$. This is finally where the coarse-graining trick comes into play $(D_0, \Delta_0) \to (D_0^k, \Delta_0^k)$.

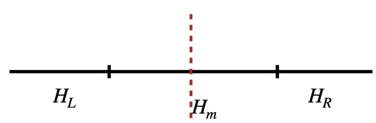
12. Pick m, q, k to obtain the desired AGSP.

□

In summary, the proof relies on the Chebyshev polynomials to construct a good AGSP in the spirit of the detectability lemma operator.

3.2 Frustrated systems

A follow-up work by Arad et al. [3] has generalized the Chebyshev-based AGSP approach to prove the area law for frustrated 1D systems. This work does not use the DL operator but instead constructs a truncated Hamiltonian $H^{(t)} = H_L^{(t)} + H_m + H_R^{(t)}$ from the terms in the original Hamiltonian H, thus can be generalized to the frustrated case. Here H_m contains the m terms around the cut and $t > 0$ is a threshold parameter above which all eigenvalues of H_L or H_R are set to t.



The motivation for this decomposition is similar to that of the the decomposition into Π_m and Π_{rest} in the proof above. The authors of [3] define the AGSP to be a (shifted and scaled) degree-ℓ Chebyshev polynomial in $H^{(t)}$: $K = f(H^{(t)})$ (note that now the polynomial applies on all terms, not only H_m). As before, the parameters m, t, and ℓ need to be chosen satisfy Lemma 3.2, but now we also require that the ground state of $H^{(t)}$ to be a good approximate of that of the original Hamiltonian. It can be shown that for sufficiently large t the spectral gap of $H^{(t)}$ is larger than that of H, and their ground states are exponentially close in t:

Theorem 3.4 (Robustness theorem [3]). *Let $|\Omega\rangle$, ε_0, γ be the unique ground state (g.s.), the g.s. energy, and the spectral gap of H. Let $|\Omega'\rangle$, ε_0' and γ' be the equivalent quantities for $H^{(t)}$. Then for $t \geq O(\varepsilon_0/\gamma^2 + 1/\gamma)$ we have*

(a) $\gamma' = \Omega(\gamma)$

(b) $\| |\Omega'\rangle - |\Omega\rangle \| \leq 2^{-\Omega(t)}$.

If H is frustration-free, their ground states are exactly the same.

Since the ground states are the same in the frustration-free (FF) case, the analysis only needs to pick appropriate t, m, ℓ to find obtain a good AGSP (Lemma 3.2). We present an overview of the proof of the FF case first. As a reminder, we want f to be an approximate of the step function $f \approx f_{step}(x)$. When t satisfies the condition in Theorem 3.4, this means we require $f(x)$ to be small whenever $\gamma \leq x \leq u$, where $u = m + 2t$ is an upper bound for $\|H^{(t)}\|$. This error is indeed the shrinking factor Δ of the AGSP and can be controller by the following lemma:

Lemma 3.5. *There is a shifted and scaled degree-ℓ Chebyshev polynomial C_ℓ such that*

$$\sqrt{\Delta} = e^{-\Omega(\ell\sqrt{\gamma/\|H^{(t)}\|})} = e^{-\Omega(\ell\sqrt{\gamma/(m+2t)})}.$$

The Schmidt rank (SR) bound for $K = C_\ell(H^{(t)})$ requires a significantly more sophisticated proof, but the idea is similar to the amortized SR analysis mentioned in step 7 in the proof of Lemma 3.3.

Lemma 3.6. *The Schmidt rank of $K = C_\ell(H^{(t)})$ (where C_ℓ is an arbitrary degree-ℓ polynomial) is bounded by $D = (d\ell)^{O(\max\{\ell/m, \sqrt{\ell}\})}$ (recall that d is the dimension of each particle).*

Setting $\ell = O(m^2)$ and $m = O(\frac{\log^2 d}{\gamma})$ and Combining Lemma 3.5, Lemma 3.6, and Lemma 3.2, one can obtain an area law for FF systems with a entanglement entropy bound of $O(\frac{\log^3 d}{\gamma})$.

For frustrated systems, one needs to choose the parameter t more carefully so that the ground state of $H^{(t)}$ is sufficiently close to that of H. Instead of applying $C_\ell(H^{(t)})$ with the same t multiple times on a good "guess" product state (as done in Lemma 3.2), the authors of [3] apply this with varying t as the state gets closer to the true ground state. The intuition is that for a fixed t, $H^{(t)}$ is a good approximate of H in the region near the ground space. Hence, the closer we are to the groud space (zero energy), the smaller t needs to be. This reduces the SR of the resultant state, establishing an area law with the same bound of $O(\frac{\log^3 d}{\gamma})$ for entanglement entropy on general frustrated 1D systems.

4 2D frustration-free area law

Due to its controllable tradeoff between the Schmidt rank and shrinking factor, Chebyshev-based AGSP gives hopes to prove 2D area laws. Indeed, ref. [4] used ideas from it to prove an area law for 2D frustration-free systems with respect to a vertical cut. At a high level, their proof finds an AGSP for each 1D strip orthogonal to the cut, then merges these AGSPs to obtain an AGSP for the entire lattice. This approach however requires better 1D AGSPs in the sense that the degree of the polynomial approximating f_{step} should have a better dependence on the approximation error. In particular, as opposed to Lemma 3.5 where the approximation error $\varepsilon = \sqrt{\Delta} = e^{-\Omega(\ell/\sqrt{n})}$, the authors of [4] construct a polynomial such that $\varepsilon = e^{-\Omega(\ell^2/n)}$. To achieve this error dependence requires an extra assumption called "local gap":

Definition 4.1 (Locally gapped Hamiltonians)**.** *Consider a $(n+1) \times L$ lattice. Let $H = \sum_{i,j} h_{ij}$, where $h_{i,j}$ is a projector acting on the qubits at $\{i, i+1\} \times \{j, j+1\}$. For any rectangular region S, let H_S be the Hamiltonian obtained by summing all terms that have support in S. Define the local gap of H to be*

$$\gamma = \min_{S \subseteq [n] \times [L+1]} gap\left(\sum_{(i,j) \in S} h_{ij}\right).$$

If $\gamma = \Omega(1)$, we say H is locally gapped.

To see why we need this assumption, we first investigate the structure of the polynomial constructed by [4]. We consider an FF Hamiltonian $H = \sum_i h_i$. The observation is that, if the terms are commuting, then the f_{step} function can be better approximated by combining Chebyshev polynomials and the *multivariate* polynomial AND. Recall that we have seen in Section 1 that for FF systems we can assume the Hamiltonian terms are projectors without loss of generality. Since the Hamiltonian terms are commuting and have eigenvalues 0 or 1, we can work in a basis in which they are simultaneously diagonal and the problem reduces to that of approximating the product of binary variables x_i which label the eigenvalues of $\Pi_i = I - h_i$. In this basis, the *univariate* step function f_{step}, which acts on the eigenvalue of the entire Hamiltonian, can be replaced by the function $\mathrm{AND}(x_1, \ldots, x_n)$. Thus, one's task is to approximate AND by a low-degree polynomial. Although a Chebyshev polynomial by itself can be a good approximator (*univariate* in the sum $x_1 + \ldots + x_n$) for AND, we need an approximator with a better error dependence as stated earlier. Ref. [4] achieved this by combining Chebyshev polynomials with so-called robust polynomials:

Lemma 4.2 (Robust polynomials)**.** *Define on the domain* $[0,1]^n$ *the function*

$$\mathrm{Rob}(x_1, \ldots, x_n) = \begin{cases} 1 & \text{if } x_1 = \ldots = x_n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

For any $\varepsilon \leq 1/10$*, there exists a degree-*$3n$ *polynomial* $\widetilde{\mathrm{Rob}}$ *with at most* 2^{5m} *monomials such that*

$$|\widetilde{\mathrm{Rob}}(x_1, \ldots, x_n) - \mathrm{Rob}(x_1, \ldots, x_n)| \leq (10\varepsilon)^n.$$

Armed with this robust polynomial approximation for AND, ref. [4] divides the Hamiltonian terms into subsets. On each subset, a Chebyshev polynomial is used to approximate the ground state projector, then $\widetilde{\mathrm{Rob}}$ is applied on the outputs of these subsets. E.g., letting the index subsets be $I_1, I_2, \ldots, I_k$ (such that $\bigcup_i I_i = [n]$) and the Chebyshev polynomial be p, we have

$$P(x) = \widetilde{\mathrm{Rob}}\left(p\left(1 - \frac{1}{|I_1|}\sum_{j\in I_1} x_j\right), p\left(1 - \frac{1}{|I_2|}\sum_{j\in I_2} x_j\right), \ldots, p\left(1 - \frac{1}{|I_k|}\sum_{j\in I_k} x_j\right)\right).$$

The following theorem gives a bound on the error of the function P in approximating AND which the authors claim cannot be further improved.

Theorem 4.3 (Optimal approximation of AND)**.** *Let* n *be a positive integer. For every real number* $\ell \in (\sqrt{n}, n)$*, there exists a polynomial* $P(x)$ *of degree* $O(\ell)$ *such that*

$$|P(x) - \mathrm{AND}(x)| = e^{-\Omega(\ell^2/n)} \quad \text{for all} \quad x \in \{0,1\}^n.$$

At this point it is worthwhile to summarize what we have achieved. We have constructed a multivariate polynomial that approximates the f_{step} that achieves a better error bound that that of Lemma 3.5 in a commuting FF Hamiltonian. To drop the commuting terms requirement, ref. [4] defines the "merge property", which asserts that $\Pi_S \Pi_T \approx \Pi_{S\cup T}$ for overlapping intervals S, T, with error decreasing exponentially in the size of the overlap region (in the commuting case $\Pi_S \Pi_T = \Pi_{S\cup T}$). With this property, they show that a polynomial approximator of f_{step} with the same error dependence can be constructed by a recursive use of the robust polynomial, with one use of the Chebyshev polynomial and the local gap property (Definition 4.1) in the base level of the recursion and the merge property to bound the error in the recursion. Indeed, they show that the local gap property implies the merge property for the following partitions of the 2D lattice

$$H_j \stackrel{\text{def}}{=} \frac{1}{2w}\left(\sum_{1\leq i<w} h_{ij} + (I - \Pi_{L,j}) + (I - \Pi_{R,j})\right) \tag{3}$$

This construction of H_j is reminiscent of the $H_L^{(t)}, H_m, H_R^{(t)}$ decomposition in the 1D proof.

This allows the authors to prove the following theorems.

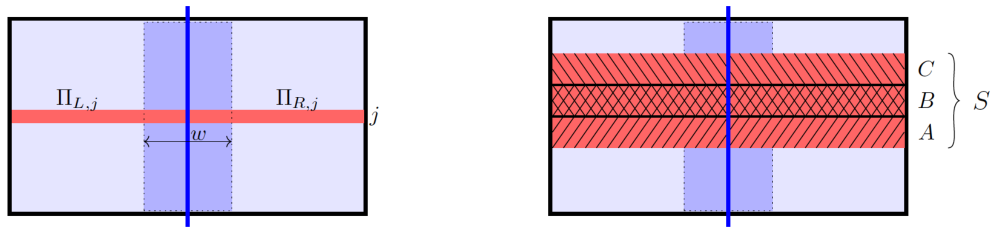


Figure 2: [4] (a) The vertical cut (blue line), region of width w (blue), and support of the operator H_j from Equation (3) (red). (b) For an interval $S \subseteq [n]$, the red region depicts the support of H_S. Given a partition $S = ABC$, the merge property asserts $\Pi_{AB}\Pi_{BC} \approx \Pi_S$ (the shading depicts AB and BC).

Theorem 4.4. *There exist real numbers $\alpha \leq n(\gamma/w)^{-1/4}$ and $\beta = (\gamma/w)^{-1/2} n^{1-O((\log n)^{-1/4})}$ such that there exists a Hermitian multivariate polynomial P in $\{H_j\}_{j=1}^n$ where*

$$P\Pi = \Pi \qquad \textit{and} \qquad \|P - \Pi\| \leq \exp\left(-\beta e^{\sqrt{\log(n)}}\right),$$

and P contains at most $(2\alpha)^\beta$ monomials of the form $(H_{j_1})^{i_1}(H_{j_2})^{i_2}\ldots(H_{j_k})^{i_k}$ with $i_1 + \ldots i_k \leq \alpha$and $k \leq \beta$.

Theorem 4.5 (Amortized SR bound)**.** *The Schmidt rank of each monomial in Theorem 4.4 is bounded by $\left(16\alpha^4 d^4 n\right)^{\frac{\alpha}{w}+\beta+wn}$.*

Combining Theorem 4.4 and Theorem 4.5, letting $K = P^{w^2}$ and $w = \Theta(n^{(\log n)^{-1/5}})$, we find a AGSP K as defined in Lemma 3.2, whereby establishing the following area law:

Theorem 4.6. *Suppose $\gamma = \Omega(1)$ and $d = O(1)$. The entanglement entropy of the ground state across the cut is at most $n^{1+O(\log^{-1/5} n)}$.*

5 Discussion

We have presented an overview of the area-law conjecture and the approximate group state projector (AGSP) technique used in the existing proofs for area laws on 1D and frustration-free 2D quantum many-body systems. Along the way, we introduced the detectability lemma and Chebyshev polynomials, which serve as tools for constructing AGSP.

General (frustrated) 2D area laws and higher dimensions remain an open problem. Numerically, an area law has been observed in a 3D superfluid system [10]. Another useful structural property of ground states is the *entanglement spread*, which roughly measures the ratio between the largest and smallest Schmidt coefficients. This quantity has an intimate connection to the communication complexity of testing bipartite ground states. Ref. [11] has shown that the entanglement spread admits an area law on any generic graph, and a sub-area law on lattices. Non-local Hamiltonians have also been studied. For example, Kuwahara and Saito [12] proved an area law for power-law interacting 1D systems.

In terms of computational complexity, Hastings [1] showed that a 1D area law implies that the ground state of a gapped Hamiltonian admits an efficient approximation (e.g. matrix product states). Thus, the problem of approximating these ground states is in **NP**. Furthermore, the algorithm in [13] shows that this problem is in fact in **P**. Interestingly, the algorithm at its core also constructs (randomized) AGSP to project a good initial state to the ground space. In this line of research, one could ask whether an area law also leads to an **NP** (or even **P**) algorithm in the 2D case.

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